Diffeomorphism Group Seminar – Introduction

Welcome!

Today: – What is this seminar about?

– Distribute topics

M^n – manifold, $M=\mathbb{R}^n$, $\mathbb{T}^n = \underbrace{S^1\times\ldots\times S^1}_{n\text{-torus}}, \ldots$

continuous smooth

$$\mathrm{Diff}^k_0(M) = \{\phi: M\to M : \exists\, \phi^{-1} \text{ and } \phi,\phi^{-1}\in C^k\} \quad k\in[0,\infty]$$

conn. to id.

$G:=\mathrm{Diff}^k(M)$ is a group.

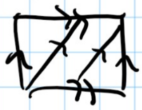
$k=n+1$ open

Homomorphisms $\mathbb{R}\to G$, $t\mapsto \phi_t: M\to M$

are flows of vector fields $X\in\mathfrak{X}(M)$ $\quad \begin{cases}\frac{d}{dt}\phi_t = X(\phi_t), \ \phi_0 = id\end{cases}$

Ex (~~rotations~~) translations $\phi_t:\mathbb{T}^n\to\mathbb{T}^n$, $\phi_t(x_1,\ldots,x_n) = (x_1+ta_1,\ldots,x_n+ta_n)$

$X = a_1\partial_{x_1}+\ldots+a_n\partial_{x_n}$



$\nearrow X$

If $\phi^1,\ldots,\phi^N$ commute (e.g., $X^1,\ldots,X^N$ with disjoint support),

then $(\phi^1,\ldots,\phi^N):\mathbb{R}^N\to G$. However, G is not abelian.

How non-abelian is it?

Normal subgroup generated by commutators.

$[G,G] = \langle\{\phi\psi\phi^{-1}\psi^{-1} : \phi,\psi\in G\}\rangle$. $\quad G$ abelian $\Leftrightarrow [G,G]=\{id\}$.

Thm (Herman + Thurston) $[G,G]=G$ (G is perfect).

Analysis $\mathbb{T}^n \Rightarrow M$ topology (foliations)

Corollary Let H be abelian, $F: G\to H$ homomorphism $\Rightarrow F=0$.

$(\ker F \supseteq [G,G]=G)$.

In particular, no homomorphism $F: G \to \mathbb{R}$. Look for quasi-morphisms $\tilde{F}: G \to \mathbb{R}$ $\left(\exists C : \forall \phi, \psi \ \left|\tilde{F}(\phi\psi) - (\tilde{F}(\phi) + \tilde{F}(\psi))\right| \leq C\right)$.

<u>Ex</u> (rotation number) $G = \mathrm{Diff}(S^1)$ $\rho(\phi)$ = rotation number of ϕ ("on average: how much ϕ rotates points).

Question: Is there $F: G \to H$ non-trivial homomorphism for some (non-abelian) H? Equivalent to say: does G have non-trivial normal subgroups. If $K \triangleleft G$, then $F: G \to G/K$.

Thm (Epstein) No. There are no non-trivial normal subgroups (G is simple).

<u>Proof</u> G perfect $\Rightarrow$ G simple (when $G = \mathrm{Diff}(M)$).

Q What happens if we restrict to diffeos preserving some geometric structures?

1) $G' = \mathrm{Diff}_{Vol}(M) = \{\phi \in G : \phi$ preserves volume on $M\}$

$G' = \mathrm{Diff}_{Symp}(M) = \{\phi \in G : \phi$ preserves symplectic form on $M\}$.

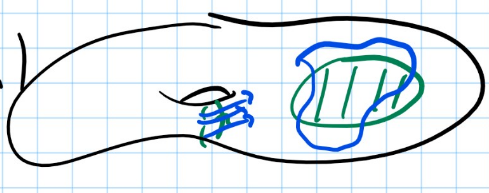
In these cases, the groups are not perfect: $[G', G'] \neq G'$

but you can describe $[G', G'] = K$ and show that K is simple.

$[G', G'] \neq G'$ because you have Flux homomorphism

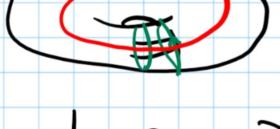
$\mathrm{Flux}: G' \to \mathrm{Hom}(H_{n-1}(M), \mathbb{R})$

$= H^{n-1}(M, \mathbb{R})$



ϕ_t flow of Vol. preserving diffeos

$\ker \mathrm{Flux}$ is simple subgroup.



ϕ_t gen. by v.f. X and Ω is the vol.

then $\phi_t^* \Omega = \Omega$ $\quad \frac{d}{dt}\phi_t^* \Omega = 0$

$ \quad\quad \phi_t^*(\mathcal{L}_X \Omega) = 0$

$\mathcal{L}_X \Omega = d\iota_X \Omega + \cancel{\iota_X d\Omega} = 0$

$\Rightarrow \iota_X \Omega$ is a $n-1$ form and it is closed $\quad \Big| \ \frac{d}{dt}\phi_t = X_t(\phi_t)$

Dates	Topics	Speakers
Sep 23	Basics on diffeos [Ch. 1]	Albert –
Sep 30	Perfect ⇒ Simple [Ch.2]	Rolf – Deborah
Oct 7	Epstein's and Herman's Thms [Ch2]	Alexander (H) –
Oct 14	Foliations, Haefliger structures [Ch2,...]	Samuel – Lauran
Oct 28	Flux Homomorphism [Ch.3, ...]	Michael –
Nov 11	Symplectic Diffeos [Ch. 4, ...]	Deborah
Nov 25	Volume-preserving Diffeos [Ch. 5]	Guy –
Dec 2	Isomorphism between Diffeo Groups [Ch.7]	Thomas –
Feb 10	(Volume-preserving) Homeos [Fathi]	Àlvaro –
Feb 24	Group of area-pres. Homeos of D^2 [CGHS]	
Mar 10	Hilbert-Smith conjecture [Pardon, Shel.]	
Mar 17	Smale conjecture on centralizers	Álvaro –