HOMOLOGICAL STABILITY

Lecture 1 Quillen's argument for homological stability of the symmetric groups.

Explain Quillen's classic (unpublished) strategy for proving homological stability of the symmetric groups (for example following [1]). Bring out clearly that the strategy is as follows: find a family of simplicial complexes W_n on which the groups G_n act such that (1) the W_n are highly (homologically) connected and (2) the stabilizers of simplices in W_n are (closely related to) the groups G_m for $m < n$. In this case the spectral sequence arising from the skeletal filtration of $W_n /\!\!/ G_n$ leads to a proof of homological stability.

Lecture 2 The general form of Quillen's argument (or the Randal-Williams–Wahl machine) and examples.

Randal-Williams and Wahl give a general form in which Quillen's argument applies in [2]. Explain some of this and show several examples (see Section 5 of the paper), e.g. braid groups, general linear groups, automorphism groups of free groups, mapping class groups.

Lecture 3 $\mathbb{E}_k$-cells: the perspective of Galatius–Kupers–Randal-Williams, part I.

(1) Give a very (!) brief review of operads and their algebras.
(2) Discuss free $\mathbb{E}_k$-algebras, pushouts of $\mathbb{E}_k$-algebras, and define cellular $\mathbb{E}_k$-algebras.
(3) As an important example, construct an $\mathbb{E}_2$-algebra that is equivalent to $\amalg_g B\Gamma_{g,1}$. Here $\Gamma_{g,1}$ is the mapping class group of a genus g surface with one marked point (or one boundary component). See [5, Section 2] or [4, Section 11.1] for different approaches.
(4) Discuss the homology of free $\mathbb{E}_k$-algebras (Dyer–Lashof operations, Browder bracket, etc.). See [3, Chapter 16] or [4, Chapter 4].

Lecture 4 GKRW, part II: Derived indecomposables and bar constructions.

The indecomposables give a way to measure the number of cells in an $\mathbb{E}_k$-algebra.

(1) Describe the (derived) indecomposables of an $\mathbb{E}_k$-algebra and explain that these may be computed by an iterated bar construction. Deduce the transfer theorem for vanishing lines. [4, Chapter 2, 7], [3, Chapter 3, 8, 13, 14].
(2) Describe how to deduce a vanishing line for the derived $\mathbb{E}_1$-indecomposables from the connectivity result for the splitting arc complex and deduce a vanishing line for derived $\mathbb{E}_2$-indecomposables. See [3, Chapter 17], [5, Section 4, 5.1].

Lecture 5 GKRW, part III: Secondary homological stability for mapping class groups.

In this talk we deduce homological stabillity and secondary homological stability with rational coefficients from the results of previous talks.

(1) State the CW-approximation and Hurewicz theorems for $\mathbb{E}_2$-algebras in simplicial $\mathbb{Q}$-modules with additional 'genus'-grading. See [4, Chapter 5].
(2) Deduce rationally homological stability and secondary homological stability. See [4, Chapter 10, 11], [5, Section 5.2].

Lecture 6 General linear groups over finite fields.

This talk is about applying the general machinery of GKRW to the homology of the groups $\mathrm{GL}_n(\mathbb{F}_q)$ with coefficients in $\mathbb{F}_p$ in the case where q is a power of p. Present [6].

Lecture 7 Classical homological stability versus $\mathbb{E}_k$-cells.

Compare the classical perspective with the method of $\mathbb{E}_k$-cells, following [7].

Lecture 8 (and possibly more) A chromatic approach to homological stability.

We'll discuss Randal-Williams' recent preprint [8].

References

[1] Alexander Kupers, *Homological Stability Minicourse Notes*, Lecture notes, 2021. Available at `https://www.utsc.utoronto.ca/people/kupers/wp-content/uploads/sites/50/homstab.pdf`.

[2] Nathalie Wahl and Oscar Randal-Williams, *Homological stability for automorphism groups*, arXiv preprint arXiv:1409.3541 [math.AT], originally submitted 11 September 2014, last revised 11 August 2017; to appear in *Advances in Mathematics*, vol. 318 (2017), pp. 534–626. Available at `https://arxiv.org/abs/1409.3541`.

[3] Søren Galatius, Alexander Kupers, and Oscar Randal-Williams, *Cellular E_k-algebras*, arXiv preprint arXiv:1805.07184 [math.AT], first posted 18 May 2018; latest version v4 posted 28 Dec 2023; to appear in *Astérisque*. Available at `https://arxiv.org/abs/1805.07184`.

[4] Søren Galatius, Alexander Kupers, and Oscar Randal-Williams, *Lecture notes on Cellular E_k-algebras*, lecture notes (unpublished). Available at `https://www.utsc.utoronto.ca/people/kupers/wp-content/uploads/sites/50/cellular-ek-algebras.pdf`.

[5] Søren Galatius, Alexander Kupers, and Oscar Randal-Williams, *E_2-cells and mapping class groups*, *Publications Mathématiques de l'IHÉS*, vol. 130 (2019), pp. 1–61. doi:10.1007/s10240-019-00107-8.

[6] Søren Galatius, Alexander Kupers, and Oscar Randal-Williams, *E_∞-Cells and General Linear Groups of Finite Fields*, *Annales Scientifiques de l'École Normale Supérieure*, vol. 57, no. 6 (2024), pp. 1845–1882. doi:10.24033/asens.2599.

[7] Oscar Randal-Williams, *Classical homological stability from the point of view of cells*, *Algebraic & Geometric Topology*, vol. 24 (2024), pp. 1691–1712. doi:10.2140/agt.2024.24.1691.

[8] Oscar Randal-Williams, *A chromatic approach to homological stability*, arXiv preprint arXiv:2508.20629 [math.AT], submitted 28 August 2025. Available at `https://arxiv.org/abs/2508.20629`.