Trace Methods and Computations

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A central theme in modern homotopy theory is the study of the stable homotopy category Sp and its structural properties are studied using algebraic K-theory. Since its inception, algebraic K-theory has been notoriously difficult to compute. This challenge has been addressed using trace methods. There exists a trace map

$$\mathrm{K} \xrightarrow{\mathrm{tr}} \mathrm{TC}$$

known as the Dennis trace map, which allows us to understand algebraic K-theory by studying TC. The Dundas-Goodwillie-McCarthy theory states that TC captures the infinitesimal behavior of K-theory by establishing that they share the same first derivative in the sense of Goodwillie calculus.

Algebraic K-theory has a motivic filtration [FS02], which raises the natural question of whether such filtrations exist for THH. Several explicit computations of Ausoni-Rognes [AR00] led them to believe that there exist such filtrations on THH of the p-completed Adams' summand which is compatible with cyclotomic structures.

The goal of this seminar is to understand these filtrations and their remarkable consequences in modern homotopy theory, majorly following [HRW22]. The structure of the seminar and syllabus is based on the 2024 Talbot seminar which one can find here. We encourage the particpant to refer to the notes from the talbot seminar and also lecture notes of Krause and Nikolaus for background on THH and TC here.

Meeting Timings: 13:30 -15:30

Meeting Venue: 707, Hans Freudenthalgebouw

Organisational Meeting: 17th October

Note that we meet on Thursdays until 7th of November and starting 20th of November we meet regularly on Wednesdays, the venue and the timing remains the same.

The main references for the topics covered in the reading seminar are the following:

- A motivic filtration on the topological cyclic homology of commutative ring spectra [HRW22].
- Topological Hochschild homology and integral p-adic Hodge theory [BMS19].
- On topological cyclic homology [NS18].

Here are a few further resources:

- Prisms and prismatic cohomology - Bhargav Bhatt and Peter Scholze (`https://www.math.uni-bonn.de/people/scholze/prisms.pdf`)
- Notes on prismatic cohomology - Kiran Kedlaya (`https://kskedlaya.org/prismatic/prismatic.html`)

- Prismatic F-gauges - Bhargav Bhatt (joint with Jacob Lurie) (https://www.math.ias.edu/~bhatt/teaching/mat549f22/lectures.pdf)
- Topological Hochschild homology in Arithmetic Geometry - Mathew Morrow (https://swc-math.github.io/aws/2019/2019MorrowNotes.pdf)
- Nilpotence and periodicity in stable homotopy theory - Doug Ravanel (https://people.math.rochester.edu/faculty/doug/mybooks/nilpb2020.pdf)
- Finite height chromatic homotopy theory - Piotr Pstrągowski (https://people.math.harvard.edu/~piotr/252y_notes.pdf)

Talks

1. **Higher Algebra (October 24, 2024)**

   Introduction to stable ∞-categories, monoidal structures, $\mathbb{E}_n$-algebras, Schwede-Shipley theorem, ∞-categorical Barr-Beck (reference : [Lur17], [Gep19]).

   Localization and completion with respect to algebra A, including concepts such as Amitsur complex and Adams tower, nilpotence, descent spectral sequence and horizontal vanishing line (reference: [MNN17; Gep19]).

2. **Introduction to THH (October 31, 2024)**

   Hochschild homology of an $\mathbb{E}_1$-algebra, cyclic bar construction, computing $\mathrm{HH}(\mathbb{F}_\mathrm{p}/\mathbb{Z})$, state the HKR theorem, and prove it in the case of polynomial rings. $\mathrm{HC}^-(\mathbb{F}_\mathrm{p}/\mathbb{Z})$, Hochschild homology as a trace and circle action on HH (reference: [NS18; HSS17; CCR+23]).

   Time permitting: THH of $\mathbb{E}_\infty$-algebras as a tensor with S^1 (see: [NS18]).

3. **Power Operations in Stable Homotopy Theory (November 7, 2024)**

   Introduction to power operations, proof of Steinberger's theorem ([BMM+86, III.2, Theorem 2.2]), universal property of Thom spectra (see: [AB18]), $\mathbb{F}_p$ as Thom spectrum, THH of Thom spectra and calculation of $\mathrm{THH}(\mathbb{F}_p) \simeq \mathbb{F}_p[u]$, $|u| = 2$ (see: Appendix [KN19]). Explain the operation σ^2 from [HW22] and interpretation of u as $p\sigma^2$.

4. **More computation of** THH **(November 20, 2024)**

   Filtered Spectra, compute the homotopy groups of $\mathrm{THH}(\mathbb{Z})/p$ and $\mathrm{THH}(l)/(p, v_1)$ following the proof in [LL23]. Using this compute THH of j_ζ and j [LL23].

5. **Synthetic Spectra (November 27, 2024)**

   MU-synthetic spectra as modules over $\mathrm{fil}^*_{\mathrm{mot}}\mathbb{S}$ in filtered spectra, synthetic analouge ν, bigraded homotopy groups, the map τ and the Adams-Novikov spectral sequence. Toda brackets and synthetic Toda brackets, shuffling relations and Toda's formula relating Toda brackets and power operations (see reference: [BHS22]).

6. **Even Filtration 1 (December 4, 2024)**

   Recall the Adams-Novikov filtration on the sphere, the HKR filtrations and BMS filtrations [BMS19]. Construction of the even filtration [HRW22], evenly faithfully flat maps including examples

   (a) $\mathbb{S} \to \mathrm{MU}$
   (b) $\mathrm{THH}(\mathbb{S}[x]) \to \mathbb{S}[x]$
   (c) $\mathrm{THH}(\mathrm{MU}) \to \mathrm{MU}$

The even filtration on an $\mathbb{E}_\infty$-ring spectrum gives an $\mathbb{E}_\infty$-rings algebra in synthetic spectra, particularly the case of $\mathbb{S}$. Compute even filtration on $\mathrm{THH}(\mathbb{F}_p[x]/\mathbb{F}_p)$.

7. **THH and Cyclotomic Structure (December 11, 2024)**

   Tate construction, dualising complex, Tate orbit lemma, S^1-action on THH, Tate diagonal, ∞-category of cyclotomic and polygonic spectra [NS18; KMN23], cyclotomic structure on THH, homotopy and Tate fixed point spectral sequence, the maps $\mathrm{can}, \varphi : \mathrm{TC}^-(A) \to \mathrm{TP}(A)$ and define $\mathrm{TC}(A)$ as equaliser.

8. **Winter Break (December 12, 2024 January 7, 2025)**

9. **Even Filtration 2 (January 8, 2025)**

   Wilson spaces, cyclotomic bases with excellent evenness properties [HRW22], even filtration for $\mathbb{E}_1$-rings following Pstrągowski [Pst23]. Even cohomology for connective rings, comparison with even filtration of Hahn-Raksit-Wilson.

10. **Motivic Spectral Sequence for $\mathrm{THH}(\mathbb{Z})^\wedge_p$ (January 15, 2025)**

    Using the eff map $\mathrm{THH}(\mathbb{S}[x]) \to \mathbb{S}[x]$ compute the motivic spectral sequence for $\mathrm{THH}(\mathbb{Z})^\wedge_p$. Steps to follow (working in the p - complete setup):

    (a) Compute $\mathrm{THH}(\mathbb{Z}/\mathbb{S}[x]^{\otimes \bullet+1})$ (refer: [KN19, Theorem 3.1] and [LW22, Corollary 4.12])

    (b) Describe the motivic spectral sequence in terms of descent object $\mathrm{THH}(\mathbb{Z}) \to \mathrm{Tot}(\mathrm{THH}(\mathbb{Z}/\mathbb{S}[x]^{\otimes \bullet+1}))$

    (c) Fully compute the resulting motivic spectral sequence, for example see [LW22, Section 5].

11. **Motivic spectral sequence for l via descent from MU (January 22, 2025)**

    (a) Motivic spectral sequence for $\mathrm{THH}(l)$ using $\mathrm{THH}(l/\mathrm{MU})$ (which is unlike the case of $\mathbb{Z}$, as there is no obvious eff map)

    (b) Compute $\mathrm{THH}(l/\mathrm{MU})$ following [HW22, Theorem 2.5.4].

    (c) Discuss the collapse of the motivic spectral sequence for $\mathrm{THH}(l)/(p, v_1)$ following [HW22, Proposition 6.1.6] (see also [AAR23, proof of Theorem 2.22]).

12. $\mathrm{TC}(\mathbb{F}_p)$ **and Redshift Phenomenon (January 29, 2025)**

    (a) The relations between the construction σ^2 and the circle action following [HW22, Appendix A.4] and [LL23, Section 3.1].

    (b) Using this to calculate homtopy groups of $\mathrm{TC}^-(\mathbb{F}_p)$ and $\mathrm{TP}(\mathbb{F}_p)$.

    (c) Prove redshift for $\mathrm{TC}^-(\mathbb{Z}_p/\mathrm{MU})$ and $\mathrm{TC}^-(l/\mathrm{MU})$, i.e. show that $L_{K(1)}\mathrm{TC}^-(\mathbb{Z}_p/\mathrm{MU}) \neq 0$ and $L_{K(2)}\mathrm{TC}^-(l/\mathrm{MU}) \neq 0$, see [HW22, Theorem 5.0.1, Corollary 5.0.2].

13. THH **and Integral p-adic Hodge theory 1 (February 5, 2025)**

    In this and the next we will discuss the arithmetic aspects of THH and its relations to integral p-adic Hodge theory following [BMS19].

    Discuss the following

    (a) Flat descent for cotangent complexes and Hochschild homology [BMS19, Section 3].

    (b) Quasisyntomic site [BMS19, Section 4].

    (c) Negative cyclic homology and de Rham cohomology [BMS19, Section 5].

    (d) THH over perfectoid base rings [BMS19, Section 6.1].

14. THH **and Integral p-adic Hodge theory 2 (February 12, 2025)**

    Continuation of the discussion on [BMS19]. Discuss the following:

    (a) THH over perfectoid base rings continued [BMS19, Section 6.2, 6.3].

(b) p-adic Nygaard complexes [BMS19, Section 7].

(c) The characteristic p-situation [BMS19, Section 8].

(d) Time permitting: Discuss the mixed-characteristic situation [BMS19, Section 9].

15. **Prismatic Cohomology of $\mathbb{Z}$ and l (February 19, 2025)**

Give a rapid introduction to the theory of prismatic cohomology, such as formal properties and key theorems, using the algebraic Tate spectral sequence [HRW22, Section 6.5] calculate the mod (p, v_1) prismatic cohomology of $\mathbb{Z}$ and the mod (p, v_1, v_2) prismatic cohomology of l (the speaker can choose one of the cases, as the two cases are analogous). Also conclude the bigraded homotopy groups of $\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TC}^-(\mathbb{Z})/(p, v_1)$ and $\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TC}^-(l)/(p, v_1, v_2)$. (For similar calculations also see [AAR23, Section 3]).

16. **Syntomic Cohomology of $\mathbb{Z}$ and l (February 26, 2025)**

Give a rapid introduction to syntomic cohomology, including formal properties and key theorems. Discuss the following, referring to [HRW22, Section 6.6]:

(a) Frobenius morphism motivic filtration.

(b) φ preserves λ_1.

(c) Use this to calculate the mod (p, v_1) syntomic cohomology of $\mathbb{Z}$.

(d) Calculate the mod (p, v_1, v_2) syntomic cohomology of l

(e) Collapsing Bockstein, duality and Lichtenbaum-Quillen theorem.

17. **Dundas-McCarthy Theorem (March 5, 2025)**

The aim of this talk is to discuss the results appearing in the thesis of Ramzi [Ram24, Section 4, 5], discuss the following:

(a) Trace theories

(b) Derivatives of a localising invariants

(c) Cyclotomic structures

(d) Endomorphisms of THH

18. **Redshift for Lubin-Tate Theory (March 12, 2025)**

The aim of the talk is to present Yuan's proof of redshift for Lubin-Tate theory, i.e. $K(E_n)$, the algebraic K-theory of n-th Lubin-Tate theory has non-trivial $T(n+1)$-localisation and the case of iterated K-theory of a field k, where k is of characteristic different than p.

19. **Land-Tamme and Purity in Localized K-Theory (March 19, 2025)**

The aim of the talk is to discuss the paper "Purity in Chromatically Localized Algebraic K-theory" [LMM+24], the topics to be covered:

(a) Recall relevant results of Land-Tamme [LT19] required for the proofs in [LMM+24].

(b) A brief history of redshift conjecture (see [AR00]).

(c) Proof of purity theorem.

(d) Discuss consequences, examples and applications.

20. **Chromatic Localizing Invariants and Descent (March 26, 2025)**

The aim of the talk is to discuss the paper "Descent and Vanishing in Chromatic Algebraic K-theory with group actions" [CMN+22], the topics to be covered:

(a) Nilpotence in equivariant homotopy theory [CMN+22, Section 3].

(b) Descent for p-groups, proof of Theorem A and B.

(c) Descent by normal bases.

(d) Swan induction theorems.

References

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